Exam Statistical Physics

Friday, April 19, 2013

The total number of points is 50. Do not forget to write your name and the student number on the first sheet. Some useful constants and integrals can be found below the last problem. Good luck.

1. **Defects in graphene**

   Graphene is a single layer of graphite where N carbon atoms form a honeycomb lattice (see Fig. 1). Discovered in 2003, it is now much studied as one of the most promising materials for future electronics. Assume that it costs an energy Δ to remove a carbon atom from a lattice site and to place it in the center of a hexagon to form a vacancy and an interstitial atom, as shown in Fig. 1.

   (a) Show that there are $N/2$ possible locations for interstitial atoms. [1 point]

   (b) Use the microcanonical ensemble to express the entropy of graphene through its total energy E. [4 points]

   (c) Find the number of interstitial atoms, M, at a temperature T: $k_B T \ll \Delta$. Assume that $N, M \gg 1$. [5 points]

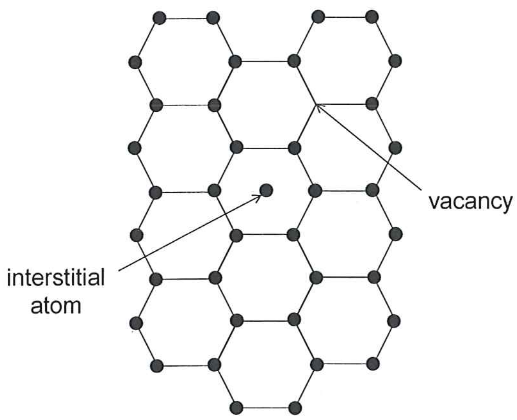

FIG. 1: Vacancies and interstitials in the honeycomb lattice of graphene.

2. **Ideal gas in gravitational field**

Consider an ideal gas of N atoms with the mass m in the gravitational field of the Earth. The gas occupies a cylinder of radius R and height H. The temperature of the gas, T, is independent of the z-coordinate.

(a) Calculate the single-atom partition function,

$$z_1 = \int \frac{d^3x d^3p}{(2\pi\hbar)^3} e^{-\beta\left(\frac{\mathbf{p}^2}{2m}+mgz\right)},$$

where g is the free-fall acceleration due to gravity and $0 < z < H$. Explain why the momentum and height distribution of atoms, $n(\mathbf{x},\mathbf{p})$, is given by

$$n(\mathbf{x},\mathbf{p}) = \frac{\lambda^3 mg}{\pi R^2 k_B T} \frac{e^{-\beta\left(\frac{\mathbf{p}^2}{2m}+mgz\right)}}{\left(1-e^{-\beta mgH}\right)},$$

where $\lambda = \frac{h}{\sqrt{2\pi m k_B T}}$ is the the thermal de Broglie wave length. The distribution is normalized by

$$\int \frac{d^3x d^3p}{(2\pi\hbar)^3} n(\mathbf{x},\mathbf{p}) = 1.$$

[2 points]

(b) Find the free energy F and the entropy S of the gas. [4 points]

(c) Find the energy E of the gas. Calculate the specific heat at constant volume per one atom, c_V, in the limits $\frac{mgH}{k_B T} \ll 1$ and $\frac{mgH}{k_B T} \gg 1$. [4 points]

3. **Phase diagram of H_2O near the triple point**

The triple point of water occurs at $T_* = 0.01$ °C and $P_* = 4.58$ mmHg. At the triple point the volume of ice $v_{ice} = 1.0907$ cm^3/g and the volume of water $v_{water} = 1.0001$ cm^3/g.

(a) Plot the (T,P) phase diagram of H_2O. Pay attention to the angles formed by the melting, boiling and sublilmation curves with the T-axis. Consider H_2O at $T = -1$ °C and a high pressure. As pressure is decreased isothermally, two successive phase transitions occur. Describe these transitions. [4 points]

(b) Find the pressure at which water transforms into ice at $T = -1$ °C. The latent heat of fusion, $L_{ice \to water} = 80$ cal/g. [6 points]

Hint: Use Clapeyron-Clausius equation. Note that the volume of the vapor is much larger than the volume of the same amount of water or ice.

4. **Compressibility of metals**

There are $\sim 2.6 \cdot 10^{22}$ conduction electrons per cm^3 of Na metal, which behave approximately as a free electron gas. From these facts,

(a) Estimate the Fermi energy of electrons, ε_F, in Na. Give the answer in eV. [4 points]

(b) Find the isothermal compressibility at zero absolute temperature,

$$\kappa = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_{T=0},$$

where V is volume and P is pressure of the electron gas. Express κ through the density of electrons n and the Fermi energy ε_F. [6 points]

Hint: Recall that $PV = \frac{2}{3}E$, where E is the total energy of the electron gas.

5. **How many photons are there in the universe?**

Imagine the universe to be a spherical cavity, with a radius of 10^{28} cm and impenetrable walls. If the temperature inside the cavity is 2.7 K, estimate the total number of photons in the universe. [10 points]

Some useful constants, conversion factors and integrals:

$\hbar = 1.1 \times 10^{-34}$ J s $= 6.6 \times 10^{-16}$ eV s, $c = 3 \times 10^{8}$ m s^{-1}, $\hbar c = 2 \times 10^{-5}$ eV cm,
$m_e = 9.1 \times 10^{-31}$ kg, $m_e c^2 = 5.1 \times 10^{5}$ eV, $k_B = \frac{1\ \mathrm{eV}}{11606\ \mathrm{K}} = 1.38 \times 10^{-23}$ J K^{-1},
1 eV $= 1.6 \times 10^{-19}$ J, 1 cal $= 4.18$ J, 1 atm $= 760$ mmHg $= 1.01 \times 10^{5}$ N m^{-2},
$a_B = \frac{\hbar^2}{m_e e^2} = 0.53$ Å, $1\text{Å} = 10^{-8}$ cm, $\mathrm{Ry} = \frac{1}{2}\frac{\hbar^2}{m_e a_B^2} = \frac{1}{2}\frac{m_e e^4}{\hbar^2} = 13.6$ eV.

$$\int_0^\infty dx x^n e^{-x} = n!, \qquad \int_{-\infty}^{\infty} \frac{dx}{\sqrt{\pi}} x^{2n} e^{-x^2} = \frac{(2n+1)!!}{2^n},$$

$$I_n = \int_0^\infty \frac{dx x^{n-1}}{e^x - 1} = (n-1)!\zeta(n), \quad I_3 \approx 2.4, \quad I_4 = \frac{\pi^4}{15}.$$